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Nilpotent adjacency matrices, random graphs and quantum random variables

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Abstract

While a number of researchers have previously investigated the relationship between graph theory and quantum probability, the current work explores a new perspective. The approach of this paper is to begin with an arbitrary graph having no previously established relationship to quantum probability and to use that graph to construct a quantum probability space in which moments of quantum random variables reveal information about the graph's structure. Given an arbitrary finite graph and arbitrary odd integer $m \geqslant 3$, fermion annihilation operators are used to construct a family of quantum random variables whose mth moments correspond to the graph's m-cycles. The approach is then generalized to recover a graph's m-cycles for any integer $m \geqslant 3$ by defining nilpotent adjacency operators in terms of null-square generators of an infinite-dimensional Abelian algebra. It is shown that ordering the vertices of a simple graph induces a canonical decomposition $\Psi = \Psi^+ + \Psi^-$ on any nilpotent adjacency operator Ψ. The work concludes with applications to Markov chains and random graphs.

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1. Introduction

Beginning with a graph $G = (V, E)$ on n vertices, the vertices of G can be associated with unit coordinate vectors in $\mathbb{R}^n$. Recalling the inner product $\langle u, v\rangle = u^\dagger v$ in $\mathbb{R}^n$, and letting A denote the adjacency matrix of G, a well-known result in graph theory states that $\langle x_0, A^k x_0\rangle$ corresponds to the number of closed k-walks based at vertex $x_0 \in V$.

Of interest here is the related problem of recovering the *k-cycles* based at any vertex x_0. This can be done for any finite graph with the methods described herein.

The algebras used in this paper were originally derived as subalgebras of Clifford algebras. Clifford algebras and quantum logic gates have been discussed in works by Li [8] and Vlasov [18]. The role of Clifford algebras in quantum computing has been considered in works by Havel and Doran [6], Matzke [10] and others.

In terms of the number of multiplications performed in the algebra, the computational complexity of enumerating the Hamiltonian cycles in a graph on n vertices is $O(n^4)$. In this context, some graph problems are moved from complexity class NP into class P [15].

Links between quantum probability and graph theory have been explored in a number of works. Hashimoto, Hora and Obata (cf [5, 12]) obtained limit theorems for increasing sequences of graphs G_n whose adjacency matrices admit a *quantum decomposition* $A_n = {A_n}^+ + {A_n}^-$. Examples include Cayley graphs, Johnson graphs and distance-regular graphs.

Obata [12] uses this approach to focus on star graphs, which are obtained by gluing together the common origins of a finite number of copies of a given graph. The adjacency matrices of star graphs admit a quantum decomposition of the form $A_n = {A_n}^+ + {A_n}^- + {A_n}^\circ$. Star graphs are of particular interest because they are related to Boolean independence in quantum probability.

Homogeneous trees are also of interest in quantum probability. These are related to the free independence of Voiculescu [19].

Comb graphs, which provide models of Bose–Einstein condensation, are related to monotone independence discovered by Lu [9] and Muraki [11]. Accardi, Ben Ghorbal and Obata [2] computed the vacuum spectral distribution of the comb graph by decomposing the adjacency matrix into a sum of monotone-independent random variables.

Another relevant work is that of Franz Lehner [7], who investigated the relationships among non-crossing partitions, creation and annihilation operators, and the cycle cover polynomial of a graph. In that work, the cycle indicator polynomials of particular digraphs are used to understand the partitioned moments and cumulants occurring in Fock spaces associated with characters of the infinite symmetric group of Bożejko and Guţă [3].

In contrast to the works cited above, the philosophy of the current work is to begin with an arbitrary finite graph and then to construct an associated algebraic probability space in which the moments of random variables reveal information about the graph's structure. The graph needs to possess no particular relationship to notions of independence or Fock spaces.

1.1. Quantum probability: operators as random variables

Provided here is a cursory review of standard concepts in quantum probability theory. The primary references for this subsection are the works of Accardi *et al* (cf [2, 1]) and Hashimoto *et al* [5].

Let $\mathcal{A}$ be a complex algebra with involution $*$ and unit 1. A positive linear $*$-functional on $\mathcal{A}$ satisfying $\varphi(1) = 1$ is called a *state* on $\mathcal{A}$. The pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space*.

If $\mathcal{A}$ is commutative, the probability space is *classical*. If $\mathcal{A}$ is non-commutative, the probability space is *quantum*.

Elements of $\mathcal{A}$ are *random variables*. A *stochastic process* is a family $(X_t) \subset \mathcal{A}$ indexed by an arbitrary set T. Given a stochastic process $X = (X_t)$, the polynomial $*$-algebra $\mathcal{P}(X)$ is a $*$-subalgebra of $\mathcal{A}$. The restriction of φ to this $*$-subalgebra gives a state φ_X on $\mathcal{P}(X)$ called the *distribution* of the process X. If $T = \{1, 2, \ldots, n\}$ is a finite set, then φ_X is called the *joint distribution* of the random variables $\{X_1, \ldots, X_n\}$.

Table 1. Realizations of commutation relations.

Type	Relation	Realized by
Boson	$\left[B^-, B^+\right] = 1$	$\lambda_n = n!$
Fermion	$\{B^-, B^+\} = 1$	$\lambda_0 = \lambda_1 = 1, \lambda_n = 0$ for $n \geqslant 2$
Free	$B^- B^+ = 1$	$\lambda_n = 1$ for all $n \geqslant 0$

Let $\Gamma = \bigoplus_{n=0}^{\infty} \mathbb{C}\phi_n$ be the Hilbert space with complete orthonormal basis $\{\phi_n\}$. Here ϕ_0 represents the 'vacuum vector'. In particular, ϕ_j is the unit vector defined by $\phi_j = (0, \ldots, \underbrace{1}_{j\text{th pos.}}, \ldots, 0)^\dagger$.

Of particular interest in quantum probability theory are interacting Fock spaces. Let $\lambda_0 = 1$, and let $\{\lambda_i\}_{1\leqslant i}$ be a sequence of non-negative real numbers such that $\lambda_m = 0 \Rightarrow \lambda_{m+1} = \lambda_{m+2} = \cdots = 0$. In the case that $\lambda_n > 0$ for all n, one defines the linear operators B^+ and B^- by

$$B^+\phi_n = \sqrt{\frac{\lambda_{n+1}}{\lambda_n}}\phi_{n+1}, \qquad n \geqslant 0 \tag{1.1}$$

and

$$B^-\phi_n = \begin{cases} \sqrt{\dfrac{\lambda_n}{\lambda_{n-1}}}\phi_{n-1}, & n \geqslant 1 \\ 0, & n = 0. \end{cases} \tag{1.2}$$

On their natural domains, B^+ and B^- are mutually adjoint and closed. Operators B^+ and B^- are called the *creation* and *annihilation operators*, respectively.

The *number operator* is defined by

$$N\phi_n = n\phi_n, \qquad n \geqslant 0. \tag{1.3}$$

Recalling the definitions of the commutator and anti-commutator, which are $[a, b] = ab - ba$ and $\{a, b\} = ab + ba$, calculation shows

$$B^+B^-\phi_0 = 0, \tag{1.4}$$

$$B^-B^+\phi_n = \frac{\lambda_{n+1}}{\lambda_n}\phi_n, \tag{1.5}$$

$$\{B^+, B^-\} = \frac{{\lambda_n}^2 + \lambda_{n+1}\lambda_{n-1}}{\lambda_n\lambda_{n-1}}, \qquad n \geqslant 1, \tag{1.6}$$

$$\left[B^+, B^-\right] = \frac{{\lambda_n}^2 - \lambda_{n+1}\lambda_{n-1}}{\lambda_n\lambda_{n-1}}, \qquad n \geqslant 1, \tag{1.7}$$

$${B^+}^n\phi_0 = \sqrt{\lambda_n}\phi_n, \qquad n \geqslant 0, \tag{1.8}$$

$${B^-}^n\phi_n = \sqrt{\lambda_n}\phi_0, \qquad n \geqslant 0. \tag{1.9}$$

Realizations of commutation relations are summarized in table 1.

When there exists some $m \geqslant 1$ such that $\lambda_m > 0$ but $\lambda_n = 0$ for all $n > m$, one defines the finite-dimensional Hilbert space

$$\Gamma = \bigoplus_{n=0}^{m} \mathbb{C}\phi_n.$$

The finite-dimensional operators B^+ and B^- are then defined in the obvious way. In this case, $B^+\phi_m = 0$.

In either the finite- or infinite-dimensional case, $(\Gamma, \{\lambda_n\}, B^+, B^-)$ is called an *interacting Fock space* associated with $\{\lambda_n\}$.

1.2. Essential graph theory: terminology and notation

The reader is referred to [20] for more graph theory. A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. Two vertices $v_i, v_j \in V$ are *adjacent* if there exists an edge $\{v_i, v_j\} \in E$.

A *k-walk* $v_0v_1 \ldots v_k$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $\{v_j, v_{j+1}\} \in E$ for each $0 \leqslant j \leqslant k-1$. A *trail* is a walk in which no edge appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-circuit* is a k-walk that is also a trail. A *k-cycle* is a closed k-walk in which no vertex other than the initial/terminal vertex appears more than once.

An edge $\{v_i, v_j\} \in E$ is said to be *incident* with the vertices v_i and v_j. Similarly, v_i and v_j are said to be *incident* with edge $\{v_i, v_j\}$. The *degree* of a vertex $v \in V$ is defined as the number of edges incident to v and is denoted $\deg(v)$.

When working with a finite graph G on n vertices, one often utilizes the *adjacency matrix A* associated with G. If the vertices are labeled $\{v_1, \ldots, v_n\}$, the adjacency matrix is defined by

$$A_{ij} = \begin{cases} 1 & \text{if} \quad v_i, v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

When A is the adjacency matrix associated with a graph G on n vertices, a well-known result of graph theory states that for any positive integer k and $1 \leqslant i \leqslant n$, the entry $(A^k)_{ii}$ is the number of closed k-walks based at vertex v_i in G.

1.3. Operators as adjacency matrices

Graphs can be interpreted as operators on Hilbert spaces. In quantum probability, bounded Hermitian operators on Hilbert spaces are quantum random variables. Similarly, quantum random variables can be interpreted as graphs.

Given an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$ associated with $\{\lambda_n\}$, the operator $A = B^+ + B^- + N$ can also be interpreted as the adjacency matrix associated with an edge-weighted graph having loops. The associated finite graph is constructed with vertex set $\{\phi_0, \phi_1, \ldots, \phi_n\}$, edges $\{\{\phi_j, \phi_{j+1}\}\}$, edge weights $\left\{\sqrt{\frac{\lambda_k}{\lambda_{k-1}}}\right\}$ and k loops based at vertex ϕ_k.

In this context, the adjacency matrix A is the sum of the upper-triangular matrix B^-, the lower-triangular matrix B^+ and the diagonal matrix N. Visualizations of graphs associated with finite-dimensional realizations of the commutation relations from table 1 appear in figures 1 and 2.

A quantum Bernoulli random variable X taking values ± 1 with equal probability $\frac{1}{2}$ admits an expression of the form

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.11}$$

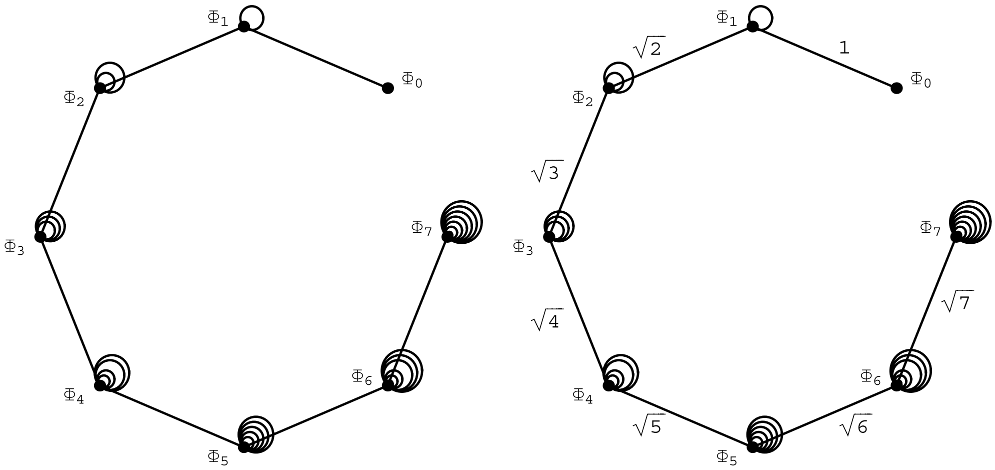


Figure 1. Graphs with adjacency matrix $B^+ + B^- + N$ for free commutation relations (left) and Boson commutation relations (right).

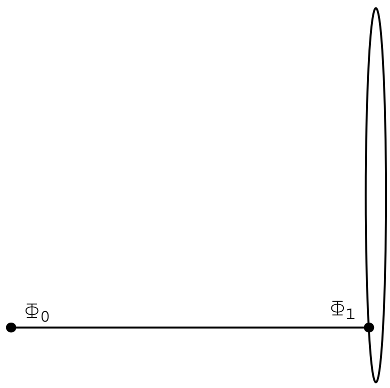


Figure 2. Graph with adjacency matrix $B^+ + B^- + N$ for Fermion commutation relations.

commonly referred to as 'quantum coin tossing'. Labeling the matrices of the right-hand side of (1.11) as f and f^+, respectively, X is decomposed into a sum of upper- and lower-triangular nilpotent matrices. It is apparent that by defining $\phi_0 = \binom{1}{0}$ and $\phi_1 = \binom{0}{1}$, one obtains

$$f^+\phi_i = \begin{cases} \phi_1 & \text{if} \quad i = 0 \\ 0 & \text{otherwise,} \end{cases} \qquad f\phi_i = \begin{cases} \phi_0 & \text{if} \quad i = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

The corresponding number operator has the form

$$f_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}$$

The matrix $A = f^+ + f + f_0$ is the adjacency matrix of the graph appearing in figure 2.

The adjacency matrix $A = B^+ + B^- + N$ corresponding to the left graph of figure 1 is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}, \tag{1.14}$$

where

$$B^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B^- = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.15}$$

and

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}. \tag{1.16}$$

2. Nilpotent adjacency operators

The approach of this paper is to begin with an arbitrary graph and use it to construct a quantum random variable whose moments reveal information about the structures contained within the graph.

Nilpotent adjacency matrix methods were first developed by the second author in [16]. Since then, the methods have been used by the authors to study random graphs [14], enumerate non-crossing partitions [13] and to reconsider the computational complexity of NP problems in the context of algebra multiplications required [15]. In addition, the second author has used nilpotent adjacency matrix methods to give a graph-theoretic construction of iterated stochastic integrals [17].

Absent from those works is a detailed inspection of nilpotent adjacency matrices from a quantum probabilistic perspective. The current work addresses this issue by constructing an infinite-dimensional probability space in which moments of quantum random variables reveal information about structures within arbitrary graphs.

2.1. Fermion adjacency operators

Adjacency operators can be defined using fermion creation or annihilation operators, and information about the cycles of length $k \equiv 1 \pmod 2$ can be recovered.

For each $n > 0$, the n-particle *fermion algebra* $\mathcal{F}_n$ is defined as the associative algebra generated by the collection $\{f_i, f_i^+\}_{1\leqslant i\leqslant n}$ satisfying the canonical anticommutation relations (CAR):

$$f_i^+ f_j^+ + f_j^+ f_i^+ = 0, \tag{2.1}$$

$$f_i f_j + f_j f_i = 0 \tag{2.2}$$

and

$$f_i f_j^+ + f_j^+ f_i = \delta_{ij}, \tag{2.3}$$

where δ_{ij} is the Kronecker delta function. The generators also satisfy the squaring rule ${f_i}^2 = {f_i^+}^2 = 0$.

Define the fermion field $\mathcal{F}$ by

$$\mathcal{F} = \bigoplus_{i=1}^{\infty} \mathcal{F}_i. \tag{2.4}$$

An arbitrary element $u \in \mathcal{F}$ has a canonical expansion of the form

$$u = \sum_{\underline{i}} u_{\underline{i}} f_{\underline{i}} + u_{\underline{i}}^+ f_{\underline{i}}^+, \tag{2.5}$$

where $\underline{i} \subset \mathbb{N}$ is a multi-index and $u_{\underline{i}}, u_{\underline{i}}^+ \in \mathbb{C}$ for each $\underline{i}$. Note that $|\underline{i}|$ denotes the cardinality of the multi-index.

Note that given an element $u \in \mathcal{F}$, the mapping $u \mapsto \tilde{u}$ defined by

$$\sum_{\underline{i}} u_{\underline{i}} f_{\underline{i}} + u_{\underline{i}}^+ f_{\underline{i}}^+ \mapsto \sum_{\underline{i}} (-1)^{|\underline{i}|(|\underline{i}|-1)/2} \left(u_{\underline{i}} f_{\underline{i}} + u_{\underline{i}}^+ f_{\underline{i}}^+\right) \tag{2.6}$$

gives an involution on $\mathcal{F}$. This involution will be referred to as the *reversion* of u.

Using parentheses to denote ordered subsets, the reversion involution satisfies the following identities:

$$f_{(i_1,i_2,\ldots,i_k)} = \widetilde{f}_{(i_k,i_{k-1},\ldots,i_1)}, \tag{2.7}$$

and

$$f^+_{(i_1,i_2,\ldots,i_k)} = \widetilde{f^+}_{(i_k,i_{k-1},\ldots,i_1)}. \tag{2.8}$$

Let $u, v \in \mathcal{F}$ with canonical expansions of the form in (2.5). An inner product is defined on $\mathcal{F}$ by

$$(u, v) = \sum_{\underline{i}} \overline{u_{\underline{i}}} v_{\underline{i}} + \overline{u_{\underline{i}}^+} v_{\underline{i}}^+. \tag{2.9}$$

This inner product on $\mathcal{F}$ induces a norm on $\mathcal{F}$ by

$$\|u\|^2 = (u, u) = \sum_{\underline{i}} |u_{\underline{i}}|^2 + \left|u_{\underline{i}}^+\right|^2. \tag{2.10}$$

Let the collection $\{x_i\}$ denote an orthonormal basis for a separable Hilbert space $\mathcal{H}$. An inner product on $\mathcal{F} \otimes \mathcal{H}$ is then defined by bilinear extension of

$$(u \otimes x_\ell, v \otimes x_k) = \delta_{\ell k}(u, v), \tag{2.11}$$

where δ_{ij} denotes the Kronecker delta function.

Note that elements of $\mathcal{F} \otimes \mathcal{H}$ are of the form $\sum_i u_i \otimes x_i$, where $u_i \in \mathcal{F}$ for each i. Consequently, this inner product defines a norm on $\mathcal{F} \otimes \mathcal{H}$ by

$$\left\langle \sum_i u_i \otimes x_i, \sum_j u_j \otimes x_j \right\rangle = \sum_i \|u_i\|^2, \tag{2.12}$$

where $\|u_i\|$ is the inner product norm on $\mathcal{F}$ defined in (2.10).

Given $u \in \mathcal{F}$, let $\overline{u}$ be defined by

$$\overline{\sum_{\underline{i}} u_{\underline{i}} f_{\underline{i}} + u^+_{\underline{i}} f^+_{\underline{i}}} = \sum_{\underline{i}} \overline{u_{\underline{i}}} f_{\underline{i}} + \overline{u^+_{\underline{i}}} f^+_{\underline{i}}. \tag{2.13}$$

The $\mathcal{F}$-inner product $\langle , \rangle : \mathcal{F} \otimes \mathcal{H} \to \mathcal{F}$ is defined on $\mathcal{F} \otimes \mathcal{H}$ by bilinear extension of

$$\langle u \otimes x_\ell, v \otimes x_k \rangle = \delta_{\ell k} \overline{u} v. \tag{2.14}$$

The 1-norm on $\mathcal{F}$ is defined by

$$\left\| \sum_{\underline{i}} u_{\underline{i}} f_{\underline{i}} + u^+_{\underline{i}} f^+_{\underline{i}} \right\|_1 = \sum_{\underline{i}} |u_{\underline{i}}| + \left|u^+_{\underline{i}}\right|. \tag{2.15}$$

Remark 2.1. Throughout the remainder of the paper, the discussion is restricted to the subalgebra of fermion annihilation operators. The results are equally valid in the subalgebra of fermion creation operators by replacing each occurrence of f_i with f_i^+. Note also that the scalar coefficients are all real valued.

Definition 2.2. *Given a graph $G = (V, E)$ on n vertices labeled by integers $1, \ldots, n$, let $\lambda : E \to \{n+1, \ldots, n+|E|\}$ be a labeling of the graph's edges. The fermion adjacency operator Φ associated with G is a bounded operator on the Hilbert space $\mathcal{F} \otimes \mathcal{H}$ defined by*

$$\Phi = \sum_{i,j} A_{ij} f_{\{j\}} f_{\{\lambda(\{v_i,v_j\})\}} |x_i\rangle\langle x_j|, \tag{2.16}$$

where A is the graph adjacency matrix defined in (1.10).

Remark 2.3. Note that vertices are also identified with basis elements x_i of $\mathcal{H}$ by the definition of the fermion adjacency operator. Consequently, there should be no confusion when referring to *vertex* x_i.

Theorem 2.4. *Let G be a graph on n vertices. Let Φ be the fermion adjacency operator associated with G. Let $m \geqslant 3$ be an integer satisfying $m \equiv 1 \pmod 2$. Then, denoting the number of distinct m-cycles based at fixed vertex x_0 by z_m,*

$$\|\langle x_0, \Phi^m x_0\rangle\|_1 = 2z_m. \tag{2.17}$$

Proof. First, it will be shown that for any positive integer m, $\langle x_0, \Phi^m x_0\rangle$ is a sum of terms in $\mathcal{F}$ corresponding to closed m-walks based at x_0. It will then be shown that as a consequence of ${f_i}^2 = 0$, all terms representing closed m-walks that revisit a vertex at an intermediate step will be removed.

First it is necessary to prove that $\langle x_i, \Phi^m x_j\rangle$ is a sum of terms in $\mathcal{F}$ corresponding to m-walks with initial vertex x_i and terminal vertex x_j. Proof is by induction on m. When $m = 1$, $\langle x_i, \Phi x_j\rangle = A_{ij} f_{\{\lambda(v_j)\}} f_{\{\lambda(\{v_i,v_j\})\}}$, and the claim is true by definition of Φ.

To simplify notation, denote by $\underline{w_{i,j}}$ the multi-index associated with the vertex/edge sequence $(v_i e_1, \ldots, e_m v_j)$ representing an m-walk from v_i to v_j. Now assuming the proposition holds for m and considering the case $m + 1$,

$$
\begin{aligned}
\Phi^{m+1} &= \Phi^m \left(\sum_{i,j} A_{ij} f_{\{j\}} f_{\{\lambda(\{v_i,v_j\})\}} |x_i\rangle\langle x_j| \right) \\
&= \left(\sum_{i,j} \sum_{\substack{m\text{-walks}\\ \underline{w_{i,j}}:v_i\to v_j}} f_{\underline{w_{i,j}}} |x_i\rangle\langle x_j| \right) \left(\sum_{k,\ell} A_{k\ell} f_{\{\ell\}} f_{\{\lambda(\{v_k,v_\ell\})\}} |x_k\rangle\langle x_\ell| \right) \\
&= \sum_{i,j} \sum_{\substack{m\text{-walks}\\ \underline{w_{i,j}}:v_i\to v_j}} f_{\underline{w_{i,j}}} \sum_{\ell} A_{j\ell} f_{\{\ell\}} f_{\{\lambda(\{v_j,v_\ell\})\}} |x_i\rangle\langle x_\ell| \\
&= \sum_{i,j} \sum_{\substack{(m+1)\text{-walks}\\ \underline{w_{i,j}}:v_i\to v_j}} f_{\underline{w_{i,j}}} |x_i\rangle\langle x_j|.
\end{aligned} \tag{2.18}
$$

Now $\langle x_i|\Phi^{m+1}|x_j\rangle = \sum_{\substack{(m+1)\text{-walks}\\ \underline{w_{i,j}}:v_i\to v_j}} f_{\underline{w_{i,j}}}$ is a sum of products of $2(m + 1)$ fermion annihilators in $\mathcal{F}$. Terms of the sum are zero in two cases.

In the first case, a vertex (or edge) is repeated at some step in the walk. Then ${f_i}^2 = 0$ appears in the product. Hence, the only nonzero terms represent $(m + 1)$-*paths* from x_i to x_j, with the possible exception that x_i is repeated once in an intermediate step. This exception occurs because the fermion annihilator associated with the initial vertex of the walk is not included in the product. When $i = j$, this fermion annihilator is acquired in the last step of the walk, and the exception is dealt with. Hence, only products associated with cycles are nonzero.

In the second case, anti-commutativity of the fermion annihilators could cause some terms to have sum zero. In order for this to happen, two terms must be products of fermions representing walks of equal length with the same initial vertex and the same terminal vertex. In addition, they must include the same vertices *and* the same edges. This is only possible if the walks either form a cycle or contain a cycle as a sub-walk. Cycles as sub-walks are eliminated by the null-square property of fermions.

It now follows that terms of $\langle x_0, \Phi^m x_0\rangle$ are nonzero only if they correspond to closed m-walks $x_0 \to x_0$ in which no vertex appears more than once (the initial vertex is only represented in the last step of the walk) and no edge is used more than once. In other words, the nonzero terms correspond to *m-cycles* $x_0 \to x_0$. All that remains to be shown is that all such m-cycles are recovered.

When $m \geqslant 3$, each m-cycle has two orientations. Because each m-cycle has two orientations, it appears in the expansion of Φ^m as the sum of two representative basis multi-vectors. Using parentheses in the multi-index to denote order of multiplication in generating the multi-vector, one multi-vector is of the form

$$
\begin{aligned}
\beta &= f_{(v_1,\lambda(\{v_0,v_1\}),\ldots,v_0,\lambda(\{v_{m-1},v_0\}))} \\
&= (-1)^{\frac{m(m-1)}{2}} f_{(v_1,\ldots,v_{m-1},v_0)} f_{(\lambda(\{v_0,v_1\}),\ldots,\lambda(\{v_{m-1},v_0\}))},
\end{aligned} \tag{2.19}
$$

and the other is of the form

$$
\begin{aligned}
f_{(v_{m-1},\lambda(\{v_0,v_{m-1}\}),\ldots,v_0,\lambda(\{v_1,v_0\}))} &= (-1)^{\frac{m(m+1)}{2}} f_{(\lambda(\{v_0,v_{m-1}\}),\ldots,\lambda(\{v_1,v_0\}))} f_{(v_{m-1},\ldots,v_1,v_0)} \\
&= (-1)^{\frac{m^2+3m-2}{2}} f_{(\lambda(\{v_0,v_{m-1}\}),\ldots,\lambda(\{v_1,v_0\}))} f_{(v_0,v_{m-1},\ldots,v_1)} \\
&= (-1)^{2m-1}\widetilde{\beta} = -\widetilde{\beta}.
\end{aligned} \tag{2.20}
$$

Here $\widetilde{\beta}$ denotes *reversion*. If $\beta = -\widetilde{\beta}$, the cycle appears with multiplicity two as an entry of Φ^m. If $\beta = \widetilde{\beta}$, the two representations sum to zero, and the cycle will not appear as an entry of Φ^m. Since β is a multi-vector indexed by a set of cardinality $2m$, $\widetilde{\beta} = (-1)^{\frac{2m(2m-1)}{2}}\beta$. Hence, $\beta = -\widetilde{\beta}$ if and only if $m(2m-1)$ is odd, i.e., $m \equiv 1 \pmod 2$.

In light of these considerations, the nonzero terms of $\langle x_0, \Phi^m x_0\rangle$ represent *all* of the m-cycles in G when $m \equiv 1 \pmod 2$. □

Lemma 2.5. *Let Φ be the fermion adjacency operator associated with a finite graph G on n vertices. Then, $\Phi^m = 0$ for all $m > n$. In other words, Φ is nilpotent.*

Proof. By construction, $\langle x_i, \Phi^m x_j\rangle$ is a sum of terms corresponding to m-walks from vertex x_i to vertex x_j in a graph on n vertices. Because ${f_i}^2 = 0$ for each i, the only nonzero terms correspond to walks with no repeated edges and no repeated vertices. □

For any fermion adjacency operator Φ and fixed basis element $x_0 \in \mathcal{H}$, the projection ρ_{x_0} is defined by

$$\rho_{x_0}\Phi = \langle x_0, \Phi x_0\rangle. \tag{2.21}$$

For $u \in \ell^2(\mathcal{F})$, the *scalar sum evaluation* of u is the homogeneous linear functional defined by

$$\varphi(u) = \varphi\left(\sum_{\underline{i}} u_{\underline{i}} f_{\underline{i}}\right) = \sum_{\underline{i}} u_{\underline{i}}. \tag{2.22}$$

For arbitrary fermion adjacency operators Φ_1, Φ_2 and arbitrary $\alpha \in \mathbb{C}$, the composition $\varphi \circ \rho_{x_0}$ satisfies

$$(\varphi \circ \rho_{x_0})(\mathbf{1}) = 1, \tag{2.23}$$

$$(\varphi \circ \rho_{x_0})(\alpha\Phi_1 + \Phi_2) = \alpha(\varphi \circ \rho_{x_0})(\Phi_1) + (\varphi \circ \rho_{x_0})(\Phi_2). \tag{2.24}$$

The collection of fermion adjacency operators associated with finite graphs generates a multiplicative semigroup $\mathcal{N}$. For every $\Phi \in \mathcal{N}$, the dual operator $\Phi^\dagger$ is defined as the transpose of Φ. Then for any $\Phi \in \mathcal{N}$, $\big(\varphi \circ \rho_{x_0}\big)(\Phi^\dagger\Phi) = 0$, i.e., the positivity requirement for states in an algebraic probability space is satisfied by $\varphi \circ \rho_{x_0}$ on the semigroup $\mathcal{N}$.

Hence, $\big(\mathcal{N}, \varphi \circ \rho_{x_0}\big)$ is considered an algebraic probability space. In this context, $\big(\varphi \circ \rho_{x_0}\big)(\Phi^m)$ is the mth moment of the quantum random variable Φ in the state $\varphi \circ \rho_{x_0}$. When m is odd, the fermion adjacency operator Φ associated with a finite graph G is a quantum random variable whose mth moment in the state $\varphi \circ \rho_{x_0}$ corresponds to the number of m-cycles based at vertex x_0 in G.

Each fermion adjacency operator is a bounded linear operator on the Hilbert space $\mathcal{F} \otimes \mathcal{H}$. The *trace* of a fermion adjacency operator Φ is defined by

$$\mathrm{Tr}(\Phi) = \sum_i \langle x_i, \Phi x_i\rangle. \tag{2.25}$$

Corollary 2.6. *Let G be a graph on n vertices. Let Φ be the fermion adjacency operator associated with G. Let $m \geqslant 3$ be an integer satisfying $m \equiv 1 \pmod 2$. Then, denoting the number of distinct m-cycles contained in G by Z_m,*

$$\|\mathrm{Tr}(\Phi^m)\|_1 = 2mZ_m. \tag{2.26}$$

Proof. This is an immediate consequence of theorem 2.4, once the possibility of unwanted cancellation of terms in the trace is ruled out.

Each m-cycle in G has m choices of base point and therefore is represented by m terms in the expansion of the trace. Given $m \equiv 1 \pmod 2$ so that the representative $2m$-vectors exist in Φ^m, summing the multi-vectors without cancellation requires equality under cyclic permutation of the vertex-edge pairs in the multi-vector. This is illustrated by

$$f_{v_2,\lambda(\{v_1,v_2\})}\cdots f_{v_0,\lambda(\{v_{m-1}v_0\})}f_{v_1,\lambda(\{v_0v_1\})} = (-1)^{2(m-1)}f_{v_1,\lambda(\{v_0v_1\})}f_{v_2,\lambda(\{v_1,v_2\})}\cdots f_{v_0,\lambda(\{v_{m-1}v_0\})}. \tag{2.27}$$

It is now clear that summing the trace entries results in no undesired cancellation since $2(m-1)$ is always even. □

2.2. *General nilpotent adjacency operators*

The fermion adjacency operator approach can be extended to recover k-cycles for any integer k by defining a different algebraic probability space.

Let $\mathcal{Z}$ be the associative infinite-dimensional algebra generated by the unit scalar $\zeta_\emptyset = 1 \in \mathbb{R}$ and the commuting nilpotents $\{\zeta_{\{i\}}\}(1 \leqslant i)$ satisfying

$$\zeta_{\{i\}}\zeta_{\{j\}} = \zeta_{\{j\}}\zeta_{\{i\}} = \begin{cases}\zeta_{\{i,j\}} & \text{if} \quad i \neq j,\\ 0 & \text{otherwise.}\end{cases} \tag{2.28}$$

The algebra $\mathcal{Z}$ is generated within the fermion field $\mathcal{F}$ by $\zeta_\emptyset = 1$ along with the set $\{\zeta_{\{i\}}\}_{1\leqslant i}$, where $\zeta_{\{i\}} = f_{2i}f_{2i+1}$. An equivalent construction uses fermion annihilation operators. Commutativity is ensured by the use of disjoint fermion pairs.

As a vector space, $\mathcal{Z}$ is spanned by unit multi-vectors of the form $\zeta_{\underline{i}} = \prod_{\iota\in\underline{i}}\zeta_{\{\iota\}}$, where $\underline{i} \subset \mathbb{N}$ is a multi-index. An arbitrary element $u \in \mathcal{Z}$ has the canonical expansion $u = \sum_{\underline{i}\subset\mathbb{N}} u_{\underline{i}}\zeta_{\underline{i}}$, where $u_{\underline{i}} \in \mathbb{R}$ for each multi-index $\underline{i}$.

The ℓ^2 norm on $\mathcal{Z}$ is then defined by

$$\|u\|^2 = \sum_{\underline{i}} {u_{\underline{i}}}^2. \tag{2.29}$$

For any $u \in \ell^2(\mathcal{Z})$, the *scalar sum evaluation* of u is the homogeneous linear functional defined by

$$\varphi(u) = \sum_{\underline{i}} u_{\underline{i}}. \tag{2.30}$$

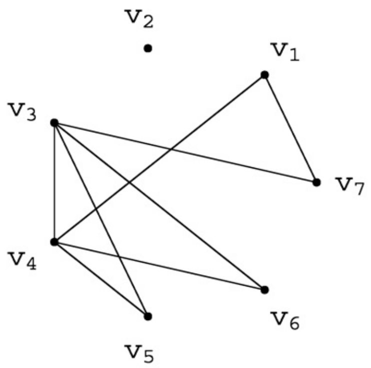
Remark 2.7. The algebra $\mathcal{Z}$ is referred to as a *zeon* algebra by Feinsilver [4]. It is the algebra referred to as $\mathcal{C}\ell_{|V|}{}^{\text{nil}}$ in Staples [17], and it is the algebra referred to as $\mathcal{N}_V$ in Schott and Staples [14].

Once again, let $\{x_i\}_{i=1}^{\infty}$ denote an orthonormal basis of a separable Hilbert space $\mathcal{H}$.

Definition 2.8. *Let $\{\zeta_{\{i\}}\}$ denote the nilpotent generators of $\mathcal{Z}$. Associated with any finite graph $G = (V, E)$ on n vertices is a nilpotent adjacency operator Ψ defined by*

$$\Psi = \sum_{i,j} A_{ij}\zeta_{\{j\}}|x_i\rangle\langle x_j|, \tag{2.31}$$

where A is the graph adjacency matrix defined in (1.10).



$$a=\begin{pmatrix} 0 & 0 & 0 & \zeta_{\{4\}} & 0 & 0 & \zeta_{\{7\}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & \zeta_{\{5\}} & \zeta_{\{6\}} & 0 \\ 0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & 0 & 0 \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3. A graph and its nilpotent adjacency operator in matrix form.

The nilpotent adjacency operator acts on the Hilbert space $\mathcal{Z} \otimes \mathcal{H}$. The dual operator $\Psi^\dagger$ is defined as the transpose of Ψ. Let the space of bounded operators on $\mathcal{Z} \otimes \mathcal{H}$ be denoted by $\mathcal{B}(\mathcal{Z} \otimes \mathcal{H})$. The nilpotent adjacency operator Ψ associated with any finite graph is then an element of $\mathcal{B}(\mathcal{Z} \otimes \mathcal{H})$.

Theorem 2.9. *Let $G = (V, E)$ be a finite graph on n vertices with associated nilpotent adjacency operator Ψ. Let $x_0 \in \mathcal{H}$ represent an arbitrary fixed vertex of G. For integer $k \geqslant 3$, let X_k denote the number of k-cycles based at x_0. Then,*

$$\varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi)x_0\rangle\bigg|_{t=0}\right) = 2X_k. \tag{2.32}$$

Proof. By definition,

$$\exp(t\Psi) = \sum_{\ell=0}^{\infty} \frac{t^\ell \Psi^\ell}{\ell!}. \tag{2.33}$$

An inductive argument proves that for any positive integer k,

$$\frac{\partial^k}{\partial t^k}(t^\ell \Psi^\ell) = \begin{cases} (\ell)_k t^{\ell-k}\Psi^\ell, & k \leqslant \ell \\ 0, & k > \ell, \end{cases} \tag{2.34}$$

where $(\ell)_k := \ell(\ell-1)\cdots(\ell-k+1)$ denotes the falling factorial. Hence,

$$\frac{\partial^k}{\partial t^k}\exp(t\Psi) = \sum_{\ell=k}^{\infty}(\ell)_k t^{\ell-k}\Psi^\ell. \tag{2.35}$$

As in the proof of theorem 2.4, entries of $\langle x_0, \Psi^\ell x_0\rangle$ correspond to ℓ-cycles based at vertex x_0. Unlike the fermion adjacency operator approach, commutativity ensures that there are no sign changes and no unwanted cancellation of terms. Hence, letting X_k denote the number of k-cycles based at an arbitrary vertex x_0 in G,

$$\varphi(\langle x_0, \Psi^k x_0\rangle) = 2X_k. \tag{2.36}$$

In light of (2.35), one finds

$$\varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi)x_0\rangle\right)\bigg|_{t=0} = 2X_k. \tag{2.37}$$

□

Denote by ρ_{x_0} the projection $\rho_{x_0}\Psi = \langle x_0, \Psi x_0\rangle$. Note that for nilpotent adjacency operators Ψ_1, Ψ_2 and for arbitrary $\alpha \in \mathbb{C}$, the composition $\varphi \circ \rho_{x_0}$ satisfies

$$\big(\varphi \circ \rho_{x_0}\big)(\mathbf{1}) = 1, \tag{2.38}$$

$$\big(\varphi \circ \rho_{x_0}\big)(\alpha\Psi_1 + \Psi_2) = \alpha\big(\varphi \circ \rho_{x_0}\big)(\Psi_1) + \big(\varphi \circ \rho_{x_0}\big)(\Psi_2). \tag{2.39}$$

More generally, the collection of nilpotent adjacency operators associated with finite graphs generates a multiplicative semigroup, $\mathcal{G}$. By construction of the nilpotent adjacency operators, $\big(\varphi \circ \rho_{x_0}\big)(\Psi^\dagger\Psi) = 0$ for all $\Psi \in \mathcal{G}$. Hence, the positivity requirement for states is satisfied, and $\big(\mathcal{G}, \varphi \circ \rho_{x_0}\big)$ is an algebraic probability space.

In this context, $\big(\varphi \circ \rho_{x_0}\big)(\Psi^m)$ is the mth moment of the quantum random variable Ψ in the state $\varphi \circ \rho_{x_0}$. It is now evident that the nilpotent adjacency operator Ψ associated with a finite graph G is a quantum random variable whose mth moment in the state $\varphi \circ \rho_{x_0}$ corresponds to the number of m-cycles based at vertex x_0 in G.

Corollary 2.10. *Let $G = (V, E)$ and Ψ be defined as in the statement of theorem 2.9. For arbitrary integer $k \geqslant 3$, let z_k denote the number of k-cycles in G. Then,*

$$\varphi\left(\frac{\partial^k}{\partial t^k}\operatorname{Tr}(\exp(t\Psi))\bigg|_{t=0}\right) = 2kz_k. \tag{2.40}$$

Define a sequence of operators $\{\Psi_n\}(n \geqslant 1)$ in $\mathcal{B}(\mathcal{Z} \otimes \mathcal{H})$ such that for each n, Ψ_n is the nilpotent adjacency operator associated with a graph on n vertices. The sequence $\{\Psi_n\}$ will be said to *weakly converge* to the operator Ψ if for each $k \geqslant 0$ and every coordinate basis vector x_0, the following equation holds:

$$\lim_{n\to\infty} \varphi\big(\big\langle x_0, {\Psi_n}^k x_0\big\rangle\big) = \varphi(\langle x_0, \Psi^k x_0\rangle). \tag{2.41}$$

Denote this convergence by $\Psi_n \xrightarrow{w} \Psi$.

Definition 2.11. *For each $n > 0$, let $G_n = (V_n, E_n)$ denote a graph on n vertices such that $V_n \subset V_{n+1}$ for each $n > 0$. The sequence $(G_n)_{n>0}$ will be referred to as a graph process.*

Theorem 2.12 (ascending chains)**.** *Let $(G_n)_{n>0}$ be a graph process, and for each $n > 0$, let $\Psi_n \in \mathcal{B}(\mathcal{Z} \otimes \mathcal{H})$ be the nilpotent adjacency operator associated with G_n. Fix vertex x_0 in G_m for some $m > 0$. For fixed integer $k \geqslant 3$, let $X_k(n)$ denote the number of k-cycles based at x_0 in G_n for each $n \geqslant m$.*

If $\exists\Psi \in \mathcal{B}(\mathcal{C} \otimes \mathcal{H})$ such that $\Psi_n \xrightarrow{w} \Psi$ as $n \to \infty$, then

$$\varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi)x_0\rangle\right)\bigg|_{t=0} = 2\lim_{n\to\infty} X_k(n). \tag{2.42}$$

Proof. Letting G_n denote the nth graph of the sequence with associated nilpotent adjacency operator Ψ_n, orthonormal basis vectors $x_1, \ldots, x_n$ of $\mathcal{H}$ are associated with the vertices of G_n by construction of Ψ_n.

Letting $X_k(n)$ denote the number of k-cycles based at an arbitrary vertex x_0 in G_n,

$$\varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi_n)x_0\rangle\right)\bigg|_{t=0} = 2X_k n. \tag{2.43}$$

Note that the convergence $\Psi_n \xrightarrow{w} \Psi$ implies

$$\lim_{n\to\infty} \varphi(\langle x_0, \exp(\Psi_n)x_0\rangle) = \varphi(\langle x_0, \exp(\Psi)x_0\rangle). \tag{2.44}$$

Hence,

$$\lim_{n\to\infty}\varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi_n)x_0\rangle\right)\Big|_{t=0} = \varphi\left(\frac{\partial^k}{\partial t^k}\langle x_0, \exp(t\Psi)x_0\rangle\right)\Big|_{t=0}$$
$$= 2\lim_{n\to\infty} X_k(n). \tag{2.45}$$

□

The nilpotent adjacency operator approach can also be applied to Markov chains and random graphs. The next two theorems illustrate these applications.

Recall that a sequence of random variables (Y_k) taking values in $S = \{1, \ldots, n\}$ is a time-homogeneous Markov chain on n states if it satisfies the Markov property

$$\mathbb{P}(Y_k = s|Y_0 = y_0, \ldots, Y_{k-1} = y_{k-1}) = \mathbb{P}(Y_k = s|Y_{k-1} = y_{k-1}) \tag{2.46}$$

for all $s, y_0, \ldots, y_{k-1} \in S$. The transition matrix of a time-homogeneous Markov chain on n states is the stochastic matrix defined by

$$M_{ij} = \mathbb{P}(X_k = j|X_{k-1} = i). \tag{2.47}$$

Identifying the states S with the vertices of a graph G, a Markov chain is a time-homogeneous random walk on G.

Theorem 2.13 (time-homogeneous random walks on finite graphs)**.** *Let M denote the transition matrix corresponding to an n-state Markov chain (i.e., time-homogeneous random walks on a graph G), and let τ denote a nilpotent stochastic operator defined by*

$$\tau = \sum_{i,j} M_{ij}\zeta_{\{j\}}|x_i\rangle\langle x_j|, \tag{2.48}$$

where each $\zeta_{\{j\}}$ is a null-square generator of $\mathcal{Z}$.

Let the state φ be defined as in (2.30), and fix a vertex x_0 of G. Then the probability that an m-step random walk on G forms an m-cycle based at x_0 is given by

$$\mathbb{P}(m\,\text{–cycle at}\, x_0) = \varphi\circ\rho_{x_0}(\tau^m). \tag{2.49}$$

Proof. To simplify notation, let the vertex sequence $\omega = (\omega_0, \ldots, \omega_{m-1})$ represent an m-cycle, i.e., ω_i is adjacent to ω_{i+1} for $0 \leqslant i \leqslant m-1$ and ω_{m-1} is adjacent to ω_0. In light of established results and keeping the Markov property in mind, it is evident that terms of $\langle x_0|\tau^m|x_0\rangle$ have the form

$$\varphi(\langle x_0|\tau^m|x_0\rangle) = \sum_{m\text{–cycles}\,\omega: x_0\to x_0} \mathbb{P}(\omega\ \text{exists}). \tag{2.50}$$

Note that because the transition matrix M is not necessarily symmetric, the graph G is assumed to be directed. Hence, no correction needs to be made for orientation. □

Let $V = \{1, 2, \ldots, n\}$ represent a fixed set of vertices for a graph G. A random graph $G = (V, E)$ is constructed by defining a collection of pairwise-independent probabilities $0 \leqslant p_{ij} \leqslant 1$ $(1 \leqslant i \neq j \leqslant n)$ such that

$$p_{ij} = \mathbb{P}((i, j) \in E). \tag{2.51}$$

In other words, p_{ij} is the probability that there exists a directed edge from vertex i to vertex j in the graph G.

Theorem 2.14 (cycles in random graphs)**.** *Consider a random directed graph $G = (V, E)$ on n vertices, corresponding to pairwise-independent edge-existence probabilities p_{ij} $(1 \leqslant i \neq j \leqslant n)$. Let ξ denote the nilpotent adjacency operator defined by*

$$\xi = \sum_{i,j} p_{ij}\zeta_{\{j\}}|x_i\rangle\langle x_j|, \tag{2.52}$$

where each $\zeta_{\{j\}}$ is a nilpotent generator of $\mathcal{Z}$.

Let the state φ be defined as in (2.30), fix a vertex x_0, and define the random variable z_m as the number of m-cycles in G based at x_0. Then,

$$\varphi \circ \rho_{x_0}(\xi^m) = \mathbb{E}(z_m). \tag{2.53}$$

*That is, ξ is a quantum random variable whose m*th *moment in the state $\varphi\circ\rho_{x_0}$ corresponds to the expected number of m-cycles occurring in the graph.*

Proof. Because the probabilities are pairwise independent,

$$\varphi(\langle x_0|\tau^m|x_0\rangle) = \sum_{m\text{-cycles }\omega:x_0\to x_0} \prod_{\ell=1}^{m} p_{w_{\ell-1}w_\ell} = \sum_{m\text{-cycles }\omega:x_0\to x_0} \mathbb{P}(w \text{ exists}). \tag{2.54}$$

Because the graph is assumed to be directed, no correction is made for cycle orientation. □

2.3. Decomposition of nilpotent adjacency operators

In the work of Hashimoto, Hora and Obata (cf [5, 12]), fixing a vertex v_0 in a finite graph induces a stratification of all the vertices by associating each vertex with the length of the shortest path linking it with v_0. This stratification is then used to define a quantum decomposition of the graph's adjacency matrix.

The nilpotent adjacency operator of a finite graph also admits a quantum decomposition as the sum of two quantum random variables. The decomposition considered here differs from that of Hashimoto *et al*.

Define δ_{ij} to be the Kronecker delta function. Let θ_{ij} denote the *ordering symbol* defined by

$$\theta_{ij} = \begin{cases} 1 & \text{if} \quad i < j \\ 0 & \text{otherwise.} \end{cases} \tag{2.55}$$

Elements Ψ^- and Ψ^+, respectively, reside in the semigroups Λ and Υ of lower- and upper-triangular nilpotent adjacency operators, which satisfy $\langle x_i, \Psi x_j\rangle = 0$ if $i \leqslant j$ and $\langle x_i, \Psi x_j\rangle = 0$ if $i \geqslant j$, respectively. These semigroups are non-Abelian; hence, Ψ^- and Ψ^+ are quantum random variables.

The canonical quantum decomposition of the nilpotent adjacency operator Ψ associated with an arbitrary finite graph is then defined by

$$\Psi = \Psi^+ + \Psi^-, \tag{2.56}$$

where

$$\langle x_i, \Psi^+ x_j\rangle = \theta_{ij}\langle x_i, \Psi x_j\rangle, \tag{2.57}$$

$$\langle x_i, \Psi^- x_j\rangle = 1 - \theta_{ij}\langle x_i, \Psi x_j\rangle. \tag{2.58}$$

Also associated with a graph $G = (V, E)$ is a *degree operator* $A_\circ$ defined by

$$\langle x_i, A_\circ x_j\rangle = \delta_{ij}\deg(x_i). \tag{2.59}$$

Recall that $\deg(x_i)$ refers to the number of edges incident with vertex x_i in G.

Assuming the graphs being considered are simple, i.e., undirected and contain no loops and no multiple edges, the degree operator is related to the nilpotent adjacency operator by

$$\varphi(\langle x_i, \Psi^2 x_i\rangle) = \langle x_i, A_\circ x_i\rangle, \qquad \forall i \geqslant 1. \tag{2.60}$$

Unlike the quantum decomposition of Hashimoto *et al* [5], this degree operator plays no role in the canonical decomposition described here.

3. Conclusion

While the interplay between graph theory and quantum probability has been a topic of investigation in a number of earlier works, this paper is the first to construct families of quantum random variables whose mth moments reveal information about cycles in arbitrary graphs and Markov chains. This work illustrates potential benefits of applying the tools of quantum probability to general problems in combinatorics and theoretical computer science.

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